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On Constructing Multiplications on Banach Spaces*

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1. INTRODUCTION

This paper deals with the general problem of determining what kinds of multiplications can be put on a Banach space to make it into a Banach algebra. It has been shown that for a unital Banach algebra, the identity must be a vertex and a point of local uniform convexity [1; 2, pp. 33–38]. This implies that many Banach spaces (e.g., Hilbert space) cannot be made into unital Banach algebras. Grabiner has shown that any separable Banach space can be made into a radical Banach algebra of power series [6] or a semisimple Banach algebra of sequences with termwise multiplication [5].

In this paper we point out that many of the problems in the theory of strictly cyclic operator algebras as studied by Embry [4], Herrero [7], and Lambert [8] are closely related to the problem of putting multiplications on Banach spaces. In this paper we prove that any Banach space with a weak-* separable dual can be made into a radical Banach algebra of power series of any finite or countable number of indeterminates, commutative or not; a radical Banach algebra of strictly lower (or strictly upper) triangular matrices; or a radical Banach algebra of Dirichlet series. We prove that given a Banach space B with a weak-* separable dual and a complemented subspace with an unconditional basis, and given an incidence algebra $I(P)$ on a countable locally finite partially-ordered set P (as defined by Doubilet, Rota and Stanley in [3, pp. 269–271]), we can give B a separately continuous multiplication so that it is algebraically isomorphic to a dense subalgebra of $I(P)$. As a corollary of this theorem, we get that any Banach space which satisfies the conditions of the theorem can be made into a Banach algebra of lower (or upper) triangular matrices.

In this paper B denotes an infinite-dimensional real or complex Banach

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space. (b_n, β_n) denotes a *generalized basis* for B, i.e., a pair of sequences $\{b_n\}_1^\infty \subset B$, $\{\beta_n\}_1^\infty \subset B^*$ satisfying:

(i) $\beta_i(b_j) = \delta_{ij}$ (the Kronecker delta)

(ii) $\{\beta_n\}$ is total over B.

If, in addition, the linear span of $\{b_n\}$ is dense in B, we have a *Markushevich basis*. A generalized basis is *normalized* if $\| b_n \| = 1$ for all n. Any separable Banach space has a Markushevich basis [9, Theorem 7, p. 116] and it has recently been proved that a Markushevich basis can be chosen which is normalized and such that $\{\| \beta_n \|\}$ is bounded [10, Theorem 1, p. 1]. B has a generalized basis if and only if B^* is weak-* separable [9, Theorem 2, p. 118]. It is easy to show that B^* is weak-* separable if and only if B^* has a countable subset which is total over B. By a *topological algebra*, we mean a Hausdorff topological vector space which is an algebra with separately continuous multiplication. By a *Banach algebra*, we mean a Banach space which is an algebra with separately continuous multiplication. Application of the uniform convergence theorem to separate continuity gives that our multiplication satisfies $\| xy \| \leqslant K \| x \| \| y \|$ for some positive constant K [12, p. 5]. So we get a Banach algebra in the strict sense (i.e., submultiplicative) under an equivalent norm or by defining a new multiplication by $x \circ y = (1/K)\, xy$. By a *lower triangular matrix* we mean an infinite matrix whose entries vanish above the diagonal. By a *strictly lower triangular matrix*, we mean an infinite matrix whose entries vanish on or above the diagonal. *Upper* and *strictly upper triangular matrices* are defined similarly.

2. Preliminary Remarks

Let $\mathfrak{A}$ be a uniformly closed algebra of bounded linear operators on a Banach space B. If, for some $e \in B$, we have $\{A(e) \mid A \in \mathfrak{A}\} = B$, then we say that e is a *strictly cyclic vector* and that $\mathfrak{A}$ is a *strictly cyclic operator algebra*. If we have, in addition, that $A(e) = 0$ implies that $A = 0$ for $A \in \mathfrak{A}$, then we say that our strictly cyclic operator algebra is *separated*. Now let $\mathfrak{A}$ be a separated, strictly cyclic operator algebra with separating strictly cyclic vector e. For $f \in B$, let A_f denote the element of $\mathfrak{A}$ such that $A_f(e) = f$. The map $T\colon \mathfrak{A} \to B$ defined by letting $T(f) = A_f$ is a Banach space isomorphism [4, Lemma 2.1, p. 444], so if we let $fg = T^{-1}(T(f)\, T(g)) = A_f A_g(e) = A_f(g)$, we make B into a Banach algebra with right identity e. Conversely, if $\mathfrak{A}$ is a Banach algebra with right identity e, the left regular representation gives us a separated strictly cyclic operator algebra. Thus many problems in the theory of separated strictly cyclic operator algebras become problems in the theory of Banach algebras. For example, Embry's result that every separated strictly cyclic operator algebra has a maximal invariant subspace [4, Theorem 3.1, p. 445] becomes a direct consequence of the fact that every modular left ideal is contained in a maximal left ideal.

It is easy to construct multiplications with identity on Banach spaces. Let $e \in B$, $e \neq 0$. Let f be a continuous linear functional with $f(e) \neq 0$. Let M be the null space of f. Then $B = M \oplus \ln\{e\}$, the direct sum of M and the linear span of $\{e\}$. If we define a separately continuous multiplication on M (e.g. we can let $xy = 0$ for $x, y \in M$), then we can define a separately continuous multiplication with identity e on B by letting $(\alpha e + m)(\beta e + n) = \alpha\beta e + \alpha n + \beta m + mn$, where $m, n \in M$ and α, β are scalars. If we require that our Banach algebras be unital, i.e., that the identity have norm 1 and that the multiplication satisfy $\| xy \| \leqslant \| x \| \| y \|$, then the problem of constructing Banach algebras with identity becomes much more difficult as we pointed out in the Introduction.

3. Main Results

THEOREM 1. *Let ϕ be a continuous injective linear map from a Banach space B onto a subalgebra of a Hausdorff topological algebra. Then the multiplication defined by letting $xy = \phi^{-1}(\phi(x)\,\phi(y))$ makes B into a Banach algebra.*

Proof. It is clear that our multiplication makes B into an algebra. We only need to prove that the multiplication is separately continuous. W.l.o.g., suppose multiplication on the left by the fixed element x is not continuous. By the closed graph theorem, there exist $y_n \in B$, with $y_n \to 0$ and $xy_n \to z \neq 0$. But by continuity, we have $\phi(z) = \lim_n \phi(xy_n) = \lim_n \phi(x)\,\phi(y_n) = \phi(x) \lim_n \phi(y_n) = 0$, a contradiction to injectiveness.

The rest of this paper is concerned with generating multiplications on Banach spaces by finding conditions such that Theorem 1 applies.

Let $\mathscr{S}$ be the space of all real or complex sequences with a multiplication defined so that it is an algebra.

DEFINITION 1. For $x = (x_1, x_2, \ldots) \in \mathscr{S}$, $x \neq 0$, we define $\operatorname{ord}(x) = \min\{n \mid x_n \neq 0\}$. We define $\operatorname{ord}(0) = \infty$.

DEFINITION 2. We say that $\mathscr{S}$ is a *shift algebra* if for all $x, y \in \mathscr{S}$, $x \neq 0 \neq y$, we have $\operatorname{ord}(xy) > \max\{\operatorname{ord}(x), \operatorname{ord}(y)\}$.

Example of a shift algebra: For $x, y \in \mathscr{S}$, let

$$xy = \left(0, x_1 y_1, \ldots, \sum_{i=1}^{n-1} x_i y_{n-i}, \ldots\right).$$

The resulting algebra is naturally isomorphic to the algebra of formal power series with zero constant coefficients.

Let π_n denote the nth coordinate linear functional for $\mathscr{S}$. We note that if $x, y \in \mathscr{S}$, then

$$\begin{aligned}\pi_n(xy) &= \pi_n([(x_1, \ldots, x_{n-1}, 0, \ldots) + (0, \ldots, 0, x_n, x_{n+1}, \ldots)] \\ &\quad \times [(y_1, \ldots, y_{n-1}, 0, \ldots) + (0, \ldots, 0, y_n, y_{n+1}, \ldots)]) \\ &= \pi_n((x_1, \ldots, x_{n-1}, 0, \ldots)(y_1, \ldots, y_{n-1}, 0, \ldots))\end{aligned}$$

by the shift property. So if we let F_n be the bilinear functional on $B \times B$ defined by $F_n(x, y) = \pi_n(xy)$, then F_n is a bilinear functional of the first $n - 1$ coordinates of x and y and we have $xy = (0, F_2(x, y), F_3(x, y), \ldots)$. We note that $\mathscr{S}$ is a complete Hausdorff topological algebra under the topology of termwise convergence. For convenience in computation, we define

$$\| F_n \| = \sup\{| F_n((x_1, \ldots, x_{n-1}, 0, \ldots), (y_1, \ldots, y_{n-1}, 0, \ldots))| \mid | x_1 | + \cdots + | x_{n-1} | = 1, | y_1 | + \cdots + | y_{n-1} | = 1\}.$$

Because of the shift property, we always have $F_1 = 0$. In the example we gave,

$$F_n(x, y) = \sum_{i=1}^{n-1} x_i y_{n-i},$$

so $\| F_n \| = 1$ for $n \geqslant 2$. In the examples of shift algebras in this paper, $\| F_n \|$ is always equal to 1 or 0.

THEOREM 2. *Let B be a Banach space with generalized basis (b_n, β_n) and let $\mathscr{S}$ be a shift algebra. Then we can pick scalars $c_n \neq 0$ so that the map $\phi: B \to \mathscr{S}$ defined by*

$$\phi(x) = (\beta_1(x)/c_1, \beta_2(x)/c_2, \ldots)$$

has the following properties:

(i) *$\phi(B)$ is a subalgebra of $\mathscr{S}$.*

(ii) *The multiplication defined by letting $xy = \phi^{-1}(\phi(x)\phi(y))$ makes B into a radical Banach algebra.*

Proof. W.l.o.g. we can assume that (b_n, β_n) is normalized. Pick $c_n \neq 0$ so that

$$\sum_{n=2}^{\infty} | c_n | \| F_n \| \left(\sum_{i=1}^{n-1} (\| \beta_i \| / | c_i |) \right)^2 < \infty. \tag{1}$$

Then we have $\phi(x)\,\phi(y) = (0, F_2(\phi(x), \phi(y)), F_3(\phi(x), \phi(y)),\ldots)$ and

$$| F_n(\phi(x), \phi(y))| \leqslant \| F_n \| \left(\sum_{i=1}^{n-1} (\| \beta_i \|/c_i) \right)^2 \| x \| \| y \| . \tag{2}$$

Because of (1) and (2),

$$z = \sum_{n=2}^{\infty} c_n F_n(\phi(x), \phi(y))\, b_n$$

is an absolutely convergent series in B, and $\phi(z) = (0, F_2(\phi(x), \phi(y)), F_3(\phi(x), \phi(y)),\ldots) = \phi(x)\,\phi(y)$. So $\phi(B)$ is a subalgebra of $\mathscr{S}$ and Theorem 1 assures us that our multiplication generates a Banach algebra.

To show that our Banach algebra is radical, we look at the extended left regular representation as defined in [12, p. 4]. Let e be the identity of the extension of B, let $x, y \in B$, and let α be a scalar. Let A_x denote left multiplication by x. Then we have

$$\begin{aligned} A_x(\alpha e + y) &= x(\alpha e + y) = \alpha x + xy \\ &= \alpha x + \sum_{n=2}^{\infty} c_n F_n(\phi(x), \phi(y))\, b_n \\ &= \alpha x + \sum_{n=2}^{N} c_n F_n(\phi(x), \phi(y))\, b_n + \sum_{n=N+1}^{\infty} c_n F_n(\phi(x), \phi(y))\, b_n \end{aligned}$$

and

$$\begin{aligned} &\left\| \sum_{n=N+1}^{\infty} c_n F_n(\phi(x), \phi(y))\, b_n \right\| \\ &\qquad \leqslant \sum_{n=N+1}^{\infty} | c_n | \| F_n \| \left(\sum_{i=1}^{n-1} \| \beta_i \| / | c_i | \right)^2 \| x \| \| y \| . \end{aligned}$$

The convergence of (1) assures us that A_x is a compact operator on the extension of B. By the Fredholm theory, all nonzero spectral values of a compact operator are eigenvalues. Since the range of A_x is contained in B, all nonzero eigenvalues must be in B. Suppose $xy = \lambda y$ with $\lambda \neq 0$. Then by the shift property we have $\mathrm{ord}(\phi(y)) = \mathrm{ord}(\lambda\phi(y)) = \mathrm{ord}(\phi(xy)) = \mathrm{ord}(\phi(x)\,\phi(y)) > \mathrm{ord}(\phi(y))$, a contradiction. So A_x, and hence x, must be quasi-nilpotent. Since x was arbitrary, B must be a radical Banach algebra.

In order to generate multiplications on Banach spaces, we want to generate shift algebras. Use of certain semigroups will help us do this.

DEFINITION 3. A countable semigroup $G = \{g_n\}_1^\infty \cup \{0\}$ is a *shift semigroup* if it satisfies:

$$0 \cdot g_n = 0 = 0^2 = g_n \cdot 0 \qquad \text{for all } n. \tag{3}$$

$$\text{If } g_i g_j = g_k, \qquad \text{then} \qquad k > \max\{i, j\}. \tag{4}$$

We note that if a countable semigroup satisfies (4), then 0 can be added so that (3) is satisfied.

THEOREM 3. *Let $G = \{g_n\}_1^\infty \cup \{0\}$ be a shift semigroup. Let us define a multiplication on the space $\mathscr{S}$ of all sequences by $(x_1, x_2, \ldots)(y_1, y_2, \ldots) = (0, z_2, z_3, \ldots)$ where*

$$z_n = \sum_{\substack{i,j \\ g_i g_j = g_n}} x_i y_j . \tag{5}$$

Then the resulting multiplication makes $\mathscr{S}$ a shift algebra.

Proof. The associativity of the semigroup gives us associativity for our multiplication. Equation (4) assures us that our multiplication is a shift.

Examples of shift semigroups:

1. Let G be the additive semigroup of positive integers with 0 added as an annihilator. The shift algebra generated is isomorphic to the algebra of formal power series with zero constant coefficients.

2. Let G be the multiplicative semigroup of the nonnegative integers with the identity omitted. The shift algebra generated is isomorphic to the algebra of formal Dirichlet series with zero constant coefficients.

Many semigroups are shift semigroups if they are sequentially ordered appropriately. The next theorem gives us necessary and sufficient conditions that this can be done.

THEOREM 4. *Let $G = \{g_n\}_1^\infty \cup \{0\}$ be a semigroup which satisfies* (3) *and such that:*

$$\textit{Each } g_n \neq 0 \textit{ has only a finite number of factorizations.} \tag{6}$$

Then the g_n can be permuted so that a shift semigroup is formed.

Proof. Note that (6) excludes the possibility of idempotents, since if $g_n = g_n^2$, then $g_n = g_n^2 = g_n^3 = \cdots$.

We say that g_i divides g_k if $g_i = g_k$ or if g_k can be expressed as a product of a finite number of g_n including g_i. Divisibility is clearly a transitive relation. It clearly will be sufficient to order the g_n sequentially so that each g_n is preceded by its proper divisors. We define a partition $\{\mathscr{P}_n\}_1^\infty$ of $G\backslash\{0\}$ inductively:

$$\mathscr{P}_1 = \{g_i \mid g_i \text{ divides } g_1\}$$

$$\mathscr{P}_{n+1} = \{g_i \mid g_i \text{ divides } g_{n+1}\}\backslash\left(\bigcup_{i=1}^{n} \mathscr{P}_i\right).$$

Condition (6) assures that each $\mathscr{P}_n$ is finite. Let $p(g_i) = n$ if $g_i \in \mathscr{P}_n$. If g_i divides g_k, then $p(g_i) \leqslant p(g_k)$. Let $l(g_i)$ equal the maximal number of factors in a factorization of g_i. Clearly, if g_i is a proper divisor of g_k, then $l(g_i) < l(g_k)$. We define a new order relation $\prec$ on $G\backslash\{0\}$ by the following: $g_i \prec g_k$ if

(i) $p(g_i) < p(g_k)$ or if

(ii) $p(g_i) = p(g_k)$ and $l(g_i) < l(g_k)$ or if

(iii) $p(g_i) = p(g_k)$ and $l(g_i) = l(g_k)$ and $i < k$.

Since the sets $\mathscr{P}_n$ are finite, $\prec$ defines an order relation which is isomorphic to the ordering on the positive integers. Since $g_i \prec g_k$, whenever g_i is a proper divisor of g_k, this completes the proof.

COROLLARY 5. *Let $G = \{g_n\}_1^\infty \cup \{0\}$ be a semigroup satisfying* (3) *and* (6). *Then G generates a multiplication on the space $\mathscr{S}$ of sequences by* (5), *and the algebra generated is isomorphic to a shift algebra by a permutation of terms.*

Proof. The existence of the multiplication in Theorem 3 depends only on the fact that each nonzero element of the semigroup can be expressed as the product of two semigroup elements in only a finite number of ways. So the multiplication is well defined for a semigroup satisfying (6). Theorem 4 assures us that an appropriate permutation of the terms will give us a shift algebra.

Corollary 5 is useful in that it shows that many important algebras are isomorphic to shift algebras and can generate multiplications for Banach spaces. In particular, if we let G be the free semigroup on some finite or countable number of noncommuting letters with 0 added as an annihilator, it is easy to see that G satisfies (3) and (6). An algebra of sequences generated from G is isomorphic to the algebra of formal power series of some number of noncommuting indeterminates. Similarly, the free semigroup on some number of commuting letters generates an algebra which is isomorphic to the algebra of formal power series of some number of commuting indeterminates. For another example, let G be the multiplicative semigroup of all infinite, strictly lower triangular matrices with at most one nonzero entry, and such that any nonzero entries must be equal to 1. The shift algebra generated is naturally isomorphic to the algebra of all infinite strictly lower triangular matrices. As a consequence of Theorems 2 and 3 and Corollary 5, we have the following theorem:

THEOREM 6. *Let B be a Banach space whose dual is weak-* separable. Then Banach algebras of the following classes can be constructed from B by the defining of multiplications:*

(i) *A radical Banach algebra of any finite or countable number of indeterminates, commutative or not.*

(ii) *A radical Banach algebra of Dirichlet series.*

(iii) *A radical Banach algebra of infinite strictly lower trangular matrices, containing all strictly lower trangular matrices with a finite number of nonzero entries.*

In Theorem 2, we looked at Banach spaces as spaces of sequences. In order to generate multiplications, we exploited the fact that if a sequence converged to zero "fast enough," it had to be in the space of sequences by the completeness property. We now wish to exploit the fact that certain bases have projections associated with them. For example, if (b_n, β_n) is an unconditional Schauder basis and if χ_K is the characteristic function of any subset of the positive integers, we can define a continuous projection P on B by letting

$$P(x) = P\left(\sum_{n=1}^{\infty} \beta_n(x)\, b_n\right) = \sum_{n=1}^{\infty} \chi_K(n)\, \beta_n(x)\, b_n$$

by [9, Theorem 3, p. 20]. We note that the image of P has an obvious unconditional basis. A similar basis projection can always be defined for a generalized basis if K is finite or cofinite in the set of positive integers. We shall extend the concept of shift semigroup to include idempotents which will correspond to certain projections. We shall generate topological algebras and again apply Theorem 1.

LEMMA 7. *Let $G = \{g_n\}_1^\infty \cup \{0\}$ be a semigroup which satisfies* (3) *and the following property:*

$$\text{\textit{Each } } g_n \neq 0 \text{\textit{ can be expressed as the product of two semigroup elements in only a finite number of ways.}} \tag{7}$$

Let $\mathscr{T}$ be the linear space of formal infinite linear combinations of g_n. Then we can define a multiplication on $\mathscr{T}$ by letting

$$\left(\sum_{n=1}^{\infty} \alpha_n g_n\right)\left(\sum_{n=1}^{\infty} \beta_n g_n\right) = \sum_{n=1}^{\infty} \left(\sum_{\substack{i,k \\ g_i g_k = g_n}} \alpha_i \beta_k\right) g_n$$

so that $\mathscr{T}$ becomes a complete, Hausdorff topological algebra under the topology of coefficientwise convergence.

Proof. If we desire strict rigor, we can define $\mathscr{T}$ to be the set of scalar functions on G with scalar value 0 on the semigroup annihilator 0. Proof that $\mathscr{T}$ is a topological algebra is essentially the same as the proof of Theorem 3. Completeness is obvious, since $\mathscr{T}$, as a topological vector space, is isomorphic to the space of sequences with the Fréchet topology.

We note that an element

$$\sum_{n=1}^{\infty} \alpha_n g_n$$

of $\mathscr{T}$ is actually a topologically convergent series as well as being a formal sum, and that any permutation of the series is topologically convergent to the same element.

DEFINITION 4. A countable semigroup $G = \{g_n\}_1^\infty \cup \{h_n\}_1^\infty \cup \{0\}$ is a *shift-idempotent semigroup* if it satisfies:

$$\{g_n\} \cup \{0\} \text{ is a subsemigroup which is a shift semigroup.} \tag{8}$$

$$\{h_n\} \cup \{0\} \text{ is a subsemigroup of orthogonal idempotents.} \tag{9}$$

For all n, m,

$$(g_n h_m = g_n \quad \text{or} \quad g_n h_m = 0) \qquad \text{and} \qquad (h_m g_n = g_n \quad \text{or} \quad h_m g_n = 0). \tag{10}$$

We want to show that Lemma 7 can be applied to a shift-idempotent semigroup $G = \{g_n\} \cup \{h_n\} \cup \{0\}$. For $n = 1, 2, \ldots$, let $L_n = \{i \mid h_n g_i = g_i\}$ and let $R_n = \{i \mid g_i h_n = g_i\}$. We note that if $h_m g_i = g_i$ and $h_n g_i = g_i$, then $(h_m h_n) g_i = g_i$. So $h_m h_n \neq 0$ and $h_m = h_n$ by (9). So the sets $\{L_n\}$ are disjoint. Similarly the sets $\{R_n\}$ are disjoint. So any g_i can be expressed as a product of two semigroup elements, one of which is an idempotent, in at most two ways. By (8), any g_i can be expressed as a product of other g_i in only a finite number of ways. Any idempotent h_i can only be factored as a power of itself.

So G satisfies condition (7) of Lemma 7 and generates a topological algebra $\mathscr{T}$ of formal infinite linear combinations of the form

$$\sum_{n=1}^{\infty} \alpha_n g_n + \sum_{n=1}^{\infty} \beta_n h_n .$$

By (8), (9), and (10) we get

$$\left(\sum_{n=1}^{\infty} \alpha_n g_n + \sum_{n=1}^{\infty} \beta_n h_n\right)\left(\sum_{n=1}^{\infty} \gamma_n g_n + \sum_{n=1}^{\infty} \delta_n h_n\right)$$

$$= \left(\sum_{n=1}^{\infty} \alpha_n g_n\right)\left(\sum_{n=1}^{\infty} \gamma_n g_n\right) + \sum_{n=1}^{\infty} \delta_n \sum_{i=1}^{\infty} \chi_{R_n}(i)\, \alpha_i g_i$$

$$+ \sum_{n=1}^{\infty} \beta_n \sum_{i=1}^{\infty} \chi_{L_n}(i)\, \gamma_i g_i + \sum_{n=1}^{\infty} \beta_n \delta_n h_n .$$

Let $\mathscr{T}_1$ denote the closed subalgebra of $\mathscr{T}$ generated by $\{g_n\}$ and let $\mathscr{T}_2$ denote the closed subalgebra of $\mathscr{T}$ generated by $\{h_n\}$. We see that $\mathscr{T} = \mathscr{T}_1 \oplus \mathscr{T}_2$ the vector space direct sum of $\mathscr{T}_1$ and $\mathscr{T}_2$, and that $\mathscr{T}_1$ is isomorphic to a shift algebra and that $\mathscr{T}_2$ is isomorphic to the algebra of all sequences with termwise multiplication.

THEOREM 8. *Let B be a Banach space whose dual is weak-* separable and which has a complemented, infinite-dimensional subspace with an unconditional Schauder basis. Let $\mathscr{T}$ be the complete topological algebra generated by a shift-idempotent semigroup $G = \{g_n\} \cup \{h_n\} \cup \{0\}$. Then there exists a continuous injective linear map $\phi\colon B \to \mathscr{T}$ which satisfies the following:*

(i) *$\phi(B)$ is a subalgebra of $\mathscr{T}$ containing all finite linear combinations of g_n and h_n.*

(ii) *The multiplication defined by letting $xy = \phi^{-1}(\phi(x)\,\phi(y))$ makes B into a Banach algebra.*

Proof. By Theorem 1, it is sufficient to prove (i). By hypothesis, $B = C \oplus D$, the direct sum of two subspaces where C has an unconditional Schauder basis (b_n, β_n) and D has a weak-* separable dual. By our introductory remarks, we know that D has a generalized basis (e_n, γ_n). We define the projections P_n, Q_n $(n = 1, 2, \ldots)$ on B by letting

$$P_n(x) = \sum_{i=1}^{\infty} \chi_{L_n}(i)\, \beta_i(x)\, b_i$$

and

$$Q_n(x) = \sum_{i=1}^{\infty} \chi_{R_n}(i)\, \beta_i(x)\, b_i$$

where χ_{L_n} and χ_{R_n} are the characteristic functions defined before.

Using Theorem 2, we choose nonzero scalars c_n such that the map $\phi_1\colon C \to \mathscr{T}_1$ defined by

$$\phi_1(x) = \sum_{n=1}^{\infty} (\beta_n(x)/c_n)\, g_n$$

induces a Banach algebra on C by the multiplication $xy = \phi_1^{-1}(\phi_1(x)\, \phi_1(y))$. We choose nonzero scalars d_n such that

$$\sum_{n=1}^{\infty} \frac{\| \gamma_n \| \| Q_n \|}{| d_n |} < \infty \tag{11}$$

$$\sum_{n=1}^{\infty} \frac{\| \gamma_n \| \| P_n \|}{| d_n |} < \infty \tag{12}$$

$$\sum_{n=1}^{\infty} \frac{\| \gamma_n \|^2}{| d_n |} < \infty. \tag{13}$$

We now define $\phi\colon B \to \mathscr{T}$ by letting

$$\phi(x) = \sum_{n=1}^{\infty} \frac{\beta_n(x)}{c_n}\, g_n + \sum_{n=1}^{\infty} \frac{\gamma_n(x)}{d_n}\, h_n .$$

ϕ is clearly continuous and $\phi(B)$ contains all finite linear combinations of g_n and h_n. We only need to prove that $\phi(B)$ is a subalgebra of $\mathscr{T}$. We have $\phi(x)\phi(y) =$

$$\left(\sum_{n=1}^{\infty} \frac{\beta_n(x)}{c_n}\, g_n + \sum_{n=1}^{\infty} \frac{\gamma_n(x)}{d_n}\, h_n\right)\left(\sum_{n=1}^{\infty} \frac{\beta_n(y)}{c_n}\, g_n + \sum_{n=1}^{\infty} \frac{\gamma_n(y)}{d_n}\, h_n\right)$$

$$= \left(\sum_{n=1}^{\infty} \frac{\beta_n(x)}{c_n}\, g_n\right)\left(\sum_{n=1}^{\infty} \frac{\beta_n(y)}{c_n}\, g_n\right) + \sum_{n=1}^{\infty} \frac{\gamma_n(y)}{d_n} \sum_{i=1}^{\infty} \chi_{R_n}(i)\, \frac{\beta_i(x)}{c_i}\, g_i$$

$$+ \sum_{n=1}^{\infty} \frac{\gamma_n(x)}{d_n} \sum_{i=1}^{\infty} \chi_{L_n}(i)\, \frac{\beta_i(y)}{c_i}\, g_i + \sum_{n=1}^{\infty} \frac{\gamma_n(x)\, \gamma_n(y)}{d_n^{\,2}}\, h_n .$$

The first of these four terms is equal to $\phi_1(z) = \phi(z)$ for some $z \in C$ by the way we chose the c_n. The second, third, and fourth terms are, respectively, equal to

$$\phi\left(\sum_{n=1}^{\infty} \frac{\gamma_n(y)}{d_n} Q_n(x)\right), \qquad \phi\left(\sum_{n=1}^{\infty} \frac{\gamma_n(x)}{d_n} P_n(y)\right),$$

and

$$\phi\left(\sum_{n=1}^{\infty} \frac{\gamma_n(x)\,\gamma_n(y)}{d_n} e_n\right).$$

Equations (11), (12), and (13) assure us of the convergence of the three series in B. So $\phi(B)$ is a subalgebra of $\mathscr{T}$ and the theorem is proved.

The condition that B contain a subspace with an unconditional basis is, of course, satisfied by any Banach space with an unconditional basis. The condition is also satisfied by any separable Banach space which contains a subspace isomorphic to c_0, since such a subspace is always complemented in a separable space [11, Theorem 4, p. 217]. The condition that the complemented subspace have an unconditional basis can be weakened in special cases, since the proof of Theorem 8 only depended on the existence of the projections P_n, Q_n.

We now show that every incidence algebra of a countable locally finite partially ordered set as defined by Doubilet, Rota and Stanley [3, pp. 269–271], is generated by a shift-idempotent semi-group.

Definition 5. We say a partially ordered set $(P, \leqslant)$ is *locally finite* if all the segments $[x, y] = \{z \in P \mid x \leqslant z \leqslant y\}$ are finite sets.

We define the incidence algebra $I(P)$ of a locally finite partially ordered set P as in [3, pp. 269–271]. $I(P)$ is the set of scalar functions on $P \times P$ which vanish off the set $\{(x, y) \mid x, y \in P, x \leqslant y\}$. Addition and scalar multiplication are defined as usual. Our multiplication $*$ is defined by letting

$$(f * g)(x, y) = \sum_{z \in P} f(x, z)\, g(z, y).$$

The local finiteness property assures us that this sum is finite. $I(P)$ is a topological algebra relative to the topology of pointwise convergence. For $x, y \in P$, $x \leqslant y$, we let δ_{xy} denote the function of $I(P)$ which takes on the value 1 at (x, y) and the value 0 elsewhere. It is shown in [3, Proposition 3.2, p. 272] that if Φ is a standard directed set of finite subsets of $P \times P$ ordered by inclusion, then the set $\{f_A \mid A \in \Phi\}$, where

$$f_A = \sum_{(x,y) \in A} f(x, y)\, \delta_{xy},$$

converges to f. So the equality

$$f_A = \sum f(x, y)\, \delta_{xy}$$

makes sense.

THEOREM 9. *Let P be a countable locally finite partially ordered set. Then the subset $G = \{\delta_{xy} \mid x, y \in P, x \leqslant y\} \cup \{0\}$ of $I(P)$ is a shift-idempotent semigroup, and the complete, Hausdorff topological algebra generated by G is isomorphic to $I(P)$.*

Proof. For δ_{xy}, $\delta_{uv} \in G$, we have $\delta_{xy}\delta_{uv} = 0$ if $y \neq u$, and $\delta_{xy}\delta_{uv} = \delta_{xu}\delta_{uv} = \delta_{xv}$ if $y = u$. The δ_{xx} are orthogonal idempotents. The subset $G' = \{\delta_{xy} \mid x, y \in P, x < y\} \cup \{0\}$ is a shift semigroup. For $\delta_{xy}, \delta_{x_0 x_1}, \ldots, \delta_{x_{n-1} x_n} \in G'$, wethave $\delta_{xy} = \delta_{x_0 x_1}\delta_{x_1 x_2} \cdots \delta_{x_{n-1} x_n}$ if and only if $x = x_0 < x_1 < \cdots < x_{n-1} < x_n = y$. By the local finiteness property, this can be done in only a finite number of ways, so that the conditions of Theorem 4 are satisfied. To show that $I(P)$ is isomorphic to the complete topological algebra generated by G, we note that

$$(f * g)(x, y) = \sum_{z \in P} f(x, z)\, g(z, y) = \sum_{\substack{(u,v),(t,w) \in P \times P \\ \delta_{uv}\delta_{tw} = \delta_{xy}}} f(u, v)\, g(t, w).$$

As a consequence of Theorems 8 and 9, we have:

THEOREM 10. *Let B be a Banach space whose dual is weak-* separable and which has a complemented, infinite-dimensional subspace with an unconditional basis. Let $I(P)$ be the incidence algebra of a countable, locally finite partially ordered set P. Then B can be given a multiplication so that it becomes a Banach algebra which is algebraically isomorphic to a subalgebra of $I(P)$ which contains all the functions of $I(P)$ with finite support.*

COROLLARY 11. *Let B be as in Theorem* 8 *and* 10. *Then B can be made into a Banach algebra of lower (upper) triangular matrices which contains all lower (upper) triangular matrices with a finite number of nonzero entries.*

Proof. Let P be the positive integers with order inverse to the natural order. $I(P)$ is algebraically isomorphic to the algebra of lower triangular matrices by the mapping which sends $f \in I(P)$ to the lower triangular matrix (α_{ij}) where $\alpha_{ij} = f(i, j)$. Similarly, the incidence algebra of the positive integers with the natural order is algebraically isomorphic to the algebra of upper triangular matrices. Application of Theorem 10 yields the corollary.

Finally, we point out some well-known results which show that our exploitation of semigroups can be carried to the fullest extent for l^1 Banach spaces. Let G be any semigroup with annihilator 0. If G does not have an annihilator, we can add one. Let $l^1(G)$ be the set of scalar functions f on G which vanish on the annihilator, have countable support and satisfy:

$$\sum_{s \in G} |f(s)| < \infty.$$

For $f, f \in l^1(G)$, we define fg by letting

$$(fg)(s) = \sum_{\substack{t,u \\ tu=s}} f(t)\, g(u)$$

for $s \neq 0$. If the nonzero elements of G form a group, this is the usual convolution.

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